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Discrete Semi-classical Orthogonal Polynomials: Generalized Meixner*

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I. At the Bar-le-Duc Conference on Orthogonal Polynomials, E. Hendriksen and H. van Rossum [1] presented a nice characterization of the class of orthogonal polynomials with quasi-orthogonal derivatives. The weight function $\rho(x)$, in the interval of orthogonality I, must satisfy a differential equation of the type

$$A(x)\,\rho'(x)+B(x)\,\rho(x)=0, \tag{1}$$

with $A(x)$ and $B(x)$ polynomials having positive leading coefficients, $\deg A \leqslant \deg B$, and $A(x)$ positive inside I.

Denoting $P_n(x)$ the nth degree orthonormal polynomial on I with respect to the weight $\rho(x)$, quasi-orthogonality of $P_n^m(x)$ means that

$$\int_I P_i^{(m)}(x)\,P_j^{(m)}(x)\,\rho_m(x)\,dx=0, \qquad \text{if} \quad |i-j|>k, \tag{2}$$

with $\rho_m(x)$ obtained from ρ, A, B, and m, and $k\neq 0$.

These polynomials $P_n(x)$, called *semi-classical* by Hendriksen and van Rossum, belong to the Freud class [2] and contain polynomials investigated in [3] that generalize the Laguerre polynomials and correspond to the weight $\rho(x)=x^{\alpha}e^{-Q(x)}$, $\alpha>-1$, $Q(x)$ being polynomials with positive leading coefficient. In that case [3] the weight is a solution of a differential equation:

$$(\sigma\rho)'=\tau\rho, \qquad \text{with} \quad \tau=\alpha+1-xQ'. \tag{3}$$

* The property of quasi-orthogonality of the derivatives of semi-classical orthogonal polynomials is extended to the discrete case for generalized Meixner polynomials.

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The aim of this paper is to give a discrete version of the property considered above as applied to an extension of the Meixner [4] class. In the discrete case, the quasi-orthogonality of a family with respect to a weight $\rho(x)$ must be interpreted, in $[0, \infty)$, as

$$\sum_{i=0}^{\infty} P_n(i)\, P_m(i)\, \rho(i) = 0, \qquad \text{for} \quad |n-m| > k \neq 0, \tag{4}$$

and will be satisfied by the difference polynomials $\Delta P_n(x) = P_n(x+1) - P_n(x)$.

III. The *classical* polynomials, in the continuous or discrete case, are characterized [5] by the following property of the positive weight $\rho(x)$: In the continuous case by $(\sigma\rho)' = \tau\rho$, in the discrete case by $\Delta(\sigma\rho) = \tau\rho$, with $\sigma(x)$ = polynomial of degree at most 2 and $\tau(x)$ = polynomial of first degree.

Additional properties are imposed at $\rho(x)$ at the end point of the support I in order to insure the existence of the moments $\mu_k = \int_0^\infty \rho(x)\, x^k\, dx$ or $\mu_k = \sum_{i=0}^\infty i^k \rho(i)$, in the discrete case in $[0, \infty)$.

It is obvious that the property of *orthogonality* of $P_n'(x)$ [or $\Delta P_n(x)$] in the classical case is a consequence of $\deg \tau = 1$ and $\deg \sigma \leqslant 2$ as we can see immediately by integration by parts,

$$\begin{aligned} 0 &= \int_0^\infty x^{m-1} P_n(x)\, \rho(x)\, \tau(x)\, dx = \int_0^\infty x^{m-1} (\sigma\rho)'\, P_n\, dx \\ &= [x^{m-1} P_n \sigma\rho]_0^\infty - \int_0^\infty \sigma\rho (x^{m-1} P_n)'\, dx \\ &= -\int_0^\infty \sigma\rho\, x^{m-1} P_n'\, dx \qquad (m < n). \end{aligned} \tag{5}$$

The last integral insures orthogonality of the P_n' with respect to $\rho_1 \equiv \sigma\rho$.

In order to generalize and to insure only *quasi-orthogonality*, we need to look at the solution of Eq. (3) with $\tau(x)$ of degree *larger* than one.

III. In the same way, let us construct in the discrete case a weight $\rho(x)$ solution of

$$\Delta(\sigma\rho) = \tau\rho \tag{6}$$

with appropriate σ and such that $\tau = \tau(x)$ be a polynomial of degree larger than one.

The difference equation (6) can be written as

$$\frac{\rho(x+1)}{\rho(x)}=\frac{\sigma(x)+\tau(x)}{\sigma(x+1)}, \tag{7}$$

and it is well known [5] that if $\sigma(x)$ and $\tau(x)$ are polynomials (assumed factorized), the solution of Eq. (7) involves Euler's Γ-function. From the asymptotic development of the ratio of two Γ-functions, namely,

$$\frac{\Gamma(a+x)}{\Gamma(x)}\simeq x^a\left[1+0\left(\frac{1}{x}\right)\right] \qquad (x\to\infty \quad a>0), \tag{8}$$

we see immediately that the degree of σ and τ must be equal in order to insure that $\lim_{x\to\infty} x^n\rho(x)=0$, for $n=0, 1, 2,...$. Let us choose $\sigma(x)=x^q$, and let us assume that $\tau(x)$, also of degree q, can be written in such a way that Eq. (7) becomes

$$\frac{\rho(x+1)}{\rho(x)}=\frac{\mu\prod_{i=1}^q(x+\gamma_i)}{(x+1)^q}. \tag{9}$$

The solution of this difference equation gives

$$\rho(x)=C\cdot\mu^x\frac{\prod_{i=1}^q\Gamma(x+\gamma_i)}{[\Gamma(x+1)]^q}. \tag{10}$$

The choices $0<\mu<1$, $\gamma_i>0$, $C>0$ insure positivity of ρ in $[0,\infty)$, $\tau(0)\neq 0$, and the existence of all moments $\mu_k=\sum_{i=0}^{\infty} i^k\rho(i)$.

Let us also remark that the choice of μ implies that the leading coefficient of $\tau(x)$ is the *negative* number $\mu-1$.

In analogy with the Meixner case let us choose $C=\prod_{i=1}^q(1/\Gamma(\gamma_i))$ so that the weight becomes, with $(\gamma)_n=\Gamma(\gamma+n)/\Gamma(\gamma)$,

$$\rho(x)=\frac{\mu^x}{[\Gamma(x+1)]^q}\prod_{i=1}^q(\gamma_i)_x; \tag{11}$$

we note $m_n^{(\gamma_i\cdots\gamma_q,\mu)}(x)$ or $m_n^{(\gamma,\mu)}(x)$ the corresponding orthogonal polynomials in $[0,\infty)$, which reduce to the Meixner polynomials [4] when $q=1$.

IV. Let us now prove the quasi-orthogonality character of the polynomials $\Delta m_n^{(\gamma,\mu)}(x)$. To simplify the typography, let us note [5]

$$p_n(i)=m_n^{(\gamma,\mu)}(i), \qquad \tau_i=\tau(i), \qquad \sigma_i=\sigma(i), \qquad \rho_i=\rho(i). \tag{12}$$

With $\tau(x)$, polynomials of degree q, the orthogonality relation gives

$$\sum_{i=0}^{\infty} i^{m-q} p_n(i)\, \tau_i \rho_i = 0, \qquad m < n, \quad q \leqslant m. \tag{13}$$

From (6), and the formula for the Δ of a product, $\Delta(f_i g_i) = f_i \Delta g_i + g_{i+1} \Delta f_i$, we obtain

$$\begin{aligned} i^{m-q} \tau_i \rho_i &= i^{m-q} \Delta(\sigma_i \rho_i) \\ &= \Delta[(i-1)^{m-q} \sigma_i \rho_i] - \sigma_i \rho_i \Delta(i-1)^{m-q}. \end{aligned} \tag{14}$$

Now (13) becomes

$$\begin{aligned} 0 &= \sum_{i=0}^{\infty} i^{m-q} p_n(i)\, \tau_i \rho_i \\ &= \sum_{i=0}^{\infty} p_n(i)\, \Delta[(i-1)^{m-q} \sigma_i \rho_i] - \sum_{i=0}^{\infty} p_n(i)\, \sigma_i \rho_i\, \Delta(i-1)^{m-q}. \end{aligned} \tag{15}$$

This last term is zero because the degree of $\sigma(x)\, \Delta(x-1)^{m-q}$ is $m-1 < n$. The first term can be written, using finite summation by parts [6], as

$$\sum_{i=0}^{\infty} f_i \Delta g_i = [f_i g_i]_0^{\infty} - \sum_{i=0}^{\infty} g_{i+1} \Delta f_i, \tag{16}$$

$$\begin{aligned} 0 &= \sum_{i=0}^{\infty} p_n(i)\, \Delta[(i-1)^{m-q} \sigma_i \rho_i] \\ &= [p_n(i)(i-1)^{m-q} \sigma_i \rho_i]_0^{\infty} - \sum_{i=0}^{\infty} i^{m-q} \sigma_{i+1} \rho_{i+1}\, \Delta p_n(i) \qquad (m-q \leqslant n-2). \end{aligned} \tag{17}$$

The boundary condition at 0 and ∞ gives zero for the first term from the choice of $\mu (\mu < 1)$ and $\sigma(x)$ $(\sigma_0 = 0)$.

The vanishing value of the last summation for all $m < n$ involves quasi-orthogonality of order $k = q-1$ with respect to the weight $\sigma_{i+1} \rho_{i+1}$, explicitly given by

$$\sigma(x+1)\, \rho(x+1) = \left(\frac{x+1}{\Gamma(x+2)} \right)^q \mu^{x+1} \prod_{i=1}^{q} (\gamma_i)_{x+1}$$

or, equivalently,

$$\sigma(x+1)\, \rho(x+1) = \frac{\mu^{x+1}}{[\Gamma(x+1)]^q} \prod_{i=1}^{q} (\gamma_i)_{x+1}.$$

V. The extension to polynomials $\Delta^k p_n(x)$ $(k > 1)$ is obvious as in the continuous case. The solution of Eq. ((7) for polynomial $\tau(x)$ contains probably the complete class of discrete orthogonal polynomials with the property shown above, but the existence of all moments and the positivity of $\rho(x)$ gives of course severe restrictions in the construction of other weights.

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